I. Frieze Patterns

<u>Defn</u>: A <u>frieze pattern</u> of height n consists of $n+2$ rows of integers

$$
\begin{array}{ccccccccccccc}
 & 1 & & 1 & & 1 & & 1 & & 1 & & \\
 & & * & & * & & * & & * & & & \\
\cdots & * & & * & & * & & * & & * & \cdots \\
 & & * & & * & & * & & * & & & \\
 & * & & * & & * & & * & & * & & \\
 & & 1 & & 1 & & 1 & & 1 & & &
\end{array}
$$

such that any diamond $\begin{array}{ccc} & B & \\ A & & C \\ & D & \end{array}$

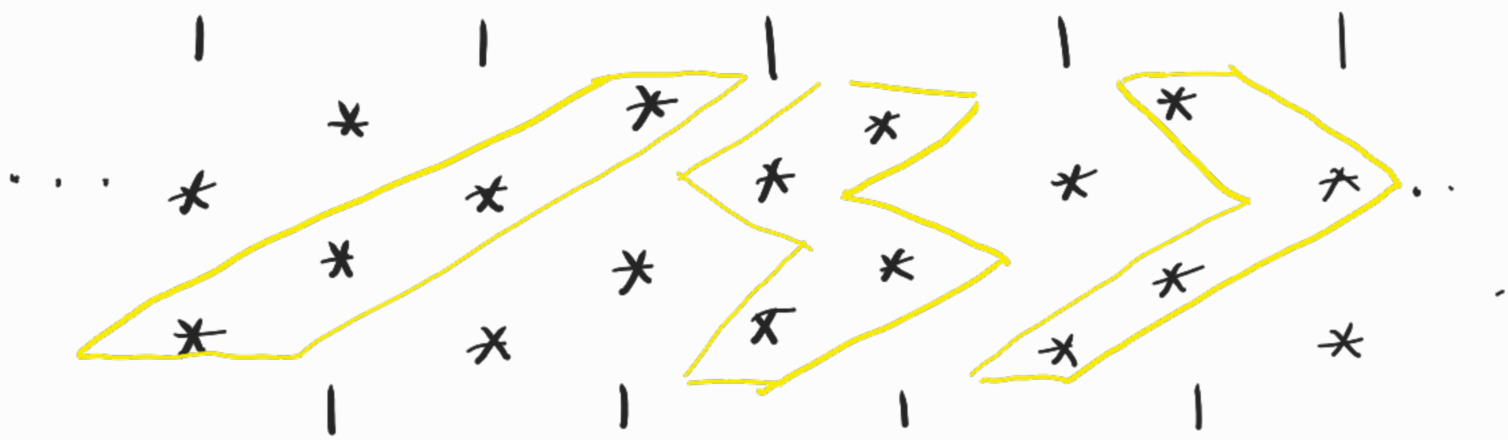
satisfies $AC - BD = 1$.

<u>Example</u>:

$$
\begin{array}{ccccccccccccccccccc}
1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & \\
 & 3 & & 1 & & 2 & & 2 & & 1 & & 3 & & 1 & & 2 & & & \\
 & & 2 & & 1 & & 3 & & 1 & & 2 & & 2 & & 1 & & 3 & & \\
 & & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 &
\end{array}
$$

- have glide symmetry
- correspond to triangulations of $(n+3)$-gon

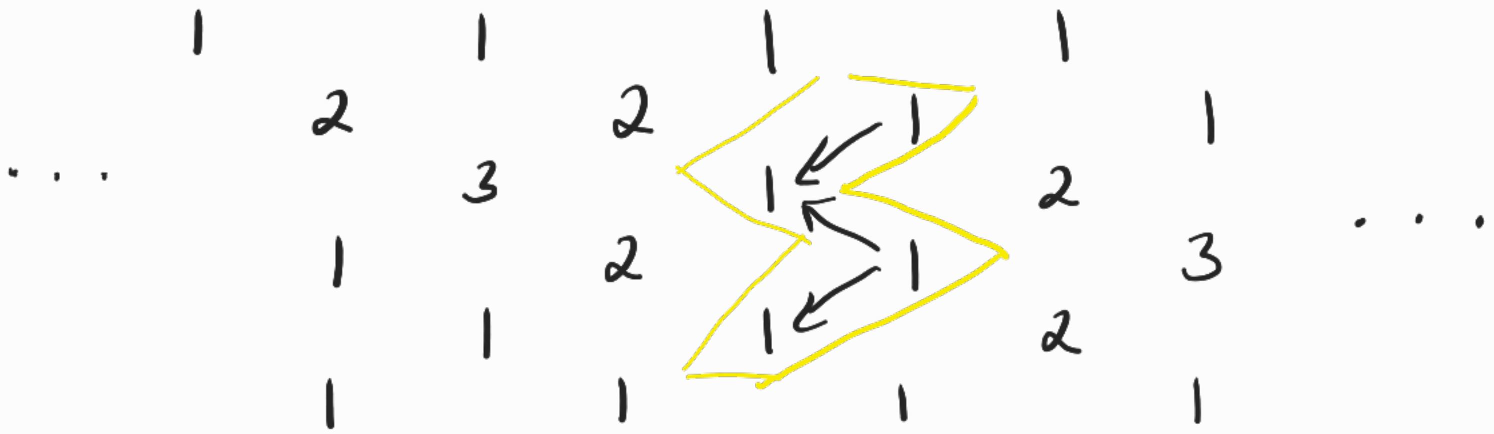
Interested in friezes from lightning bolts,
one element from each row s.t. adjacent elements are from the same diamond



Prop: A frieze is determined by the entries in any lightning bolt.

pf: Can continue "completing diamonds"

Note: Most fillings of a l.b. will lead to non-integer entries. But miraculously, if we fill with all 1's, we get a frieze!



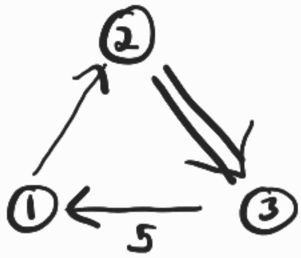
Very elegant proof via cluster algebras

II. Quivers

To construct a cluster algebra, we first need the initial data of a quiver.

Defn: A quiver is finite oriented graph with no loops or 2-cycles (multiple edges allowed)

Example 1:

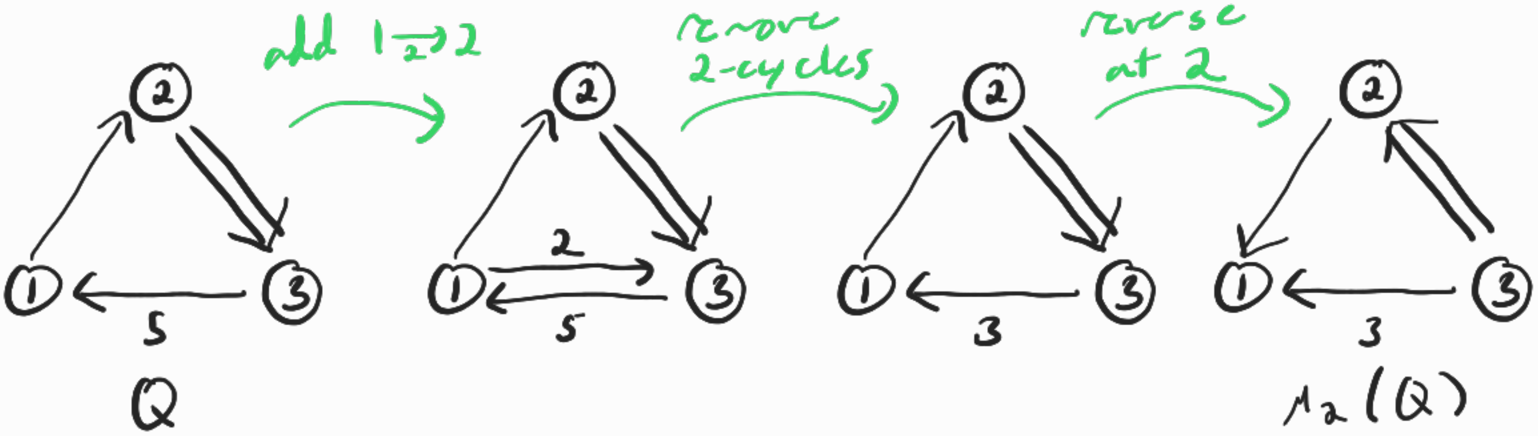


$Q \longrightarrow \mu_j(Q)$

Defn: To mutate a quiver at vertex j:

- For each directed path $i \xrightarrow{r} j \xrightarrow{s} k$, add an arrow $i \xrightarrow{rs} k$ (cancelling opposite arrows)
- Reverse all arrows incident to j

Example 2: Take Q from Ex 1. Then $\mu_2(Q)$ is



Example 3: A source (resp. sink) vertex has all incident arrows pointing away from (resp. toward) it

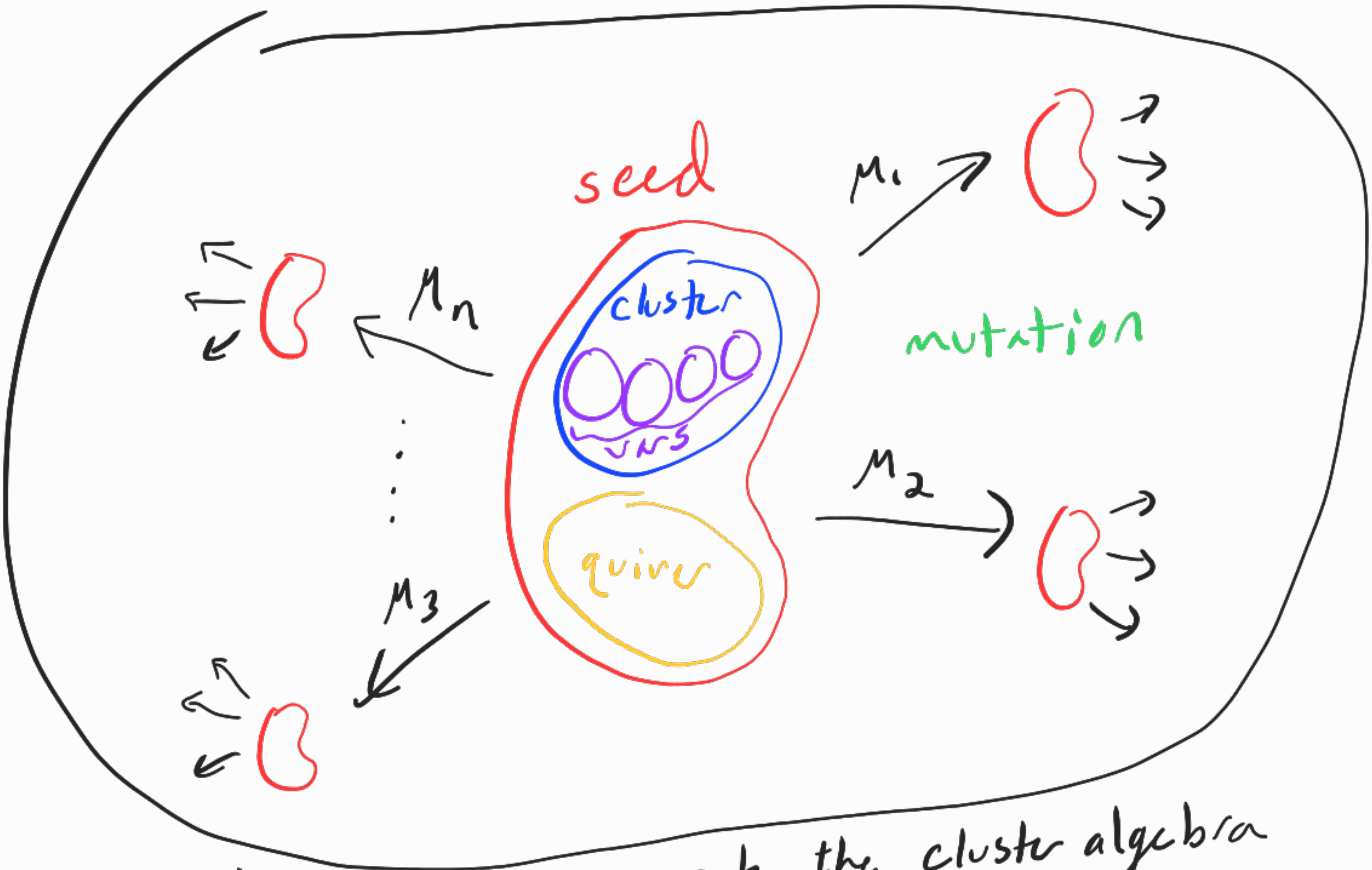
If we mutate Q at a source j, then $\mu_j(Q)$ is obtained by simply reversing arrows at j (similar for sinks)

Defn: We can add a set of frozen vertices, drawn as squares, that we are not allowed to mutate at. We then say the quiver is an ice quiver.

Defn: The rank of a quiver is the number of mutable vertices

III Cluster Algebras

We add variables to each vertex of the quiver which also mutate.



all cluster vars generate the cluster algebra

Defn. A **seed** $(\underline{x}, Q)$ consists of

(i) a cluster $\underline{x} = (x_1, \ldots, x_m)$ of "algebraically independent" variables

- $x_1, \ldots, x_n$ are cluster variables (at mutable vertices)
- $x_{n+1}, \ldots, x_m$ are coefficient vars (at frozen vertices)

(ii) a quiver Q of rank n with $m-n$ frozens

Defn: The <u>seed mutation</u> μ_j at j transforms $(\underline{x}, Q)$ into $\mu_j(\underline{x}, Q) = (\underline{x}', Q')$ given by

- $Q' = \mu_j(Q)$
- $\underline{x}' = (x_1', \dots, x_n')$ where $x_k' = x_k$ for $k \neq j$, and $\boxed{x_j x_j' = \prod_{i \to j} x_i + \prod_{j \to k} x_k}$ "exchange relation"

Example 4

$$\begin{tikzcd}[column sep=small] & x_2 \arrow[dr, shift left] \arrow[dr, shift right] & \\ x_1 \arrow[ur] & & x_3 \arrow[ll, "5"] \end{tikzcd} \quad \xrightarrow[x_2 x_2' = x_1 + x_3^2]{\mu_2} \quad \begin{tikzcd}[column sep=small] & \frac{x_1 + x_3^2}{x_2} \arrow[dr, shift left] \arrow[dr, shift right] & \\ x_1 \arrow[ur] & & x_3 \arrow[ll, "3"] \end{tikzcd}$$

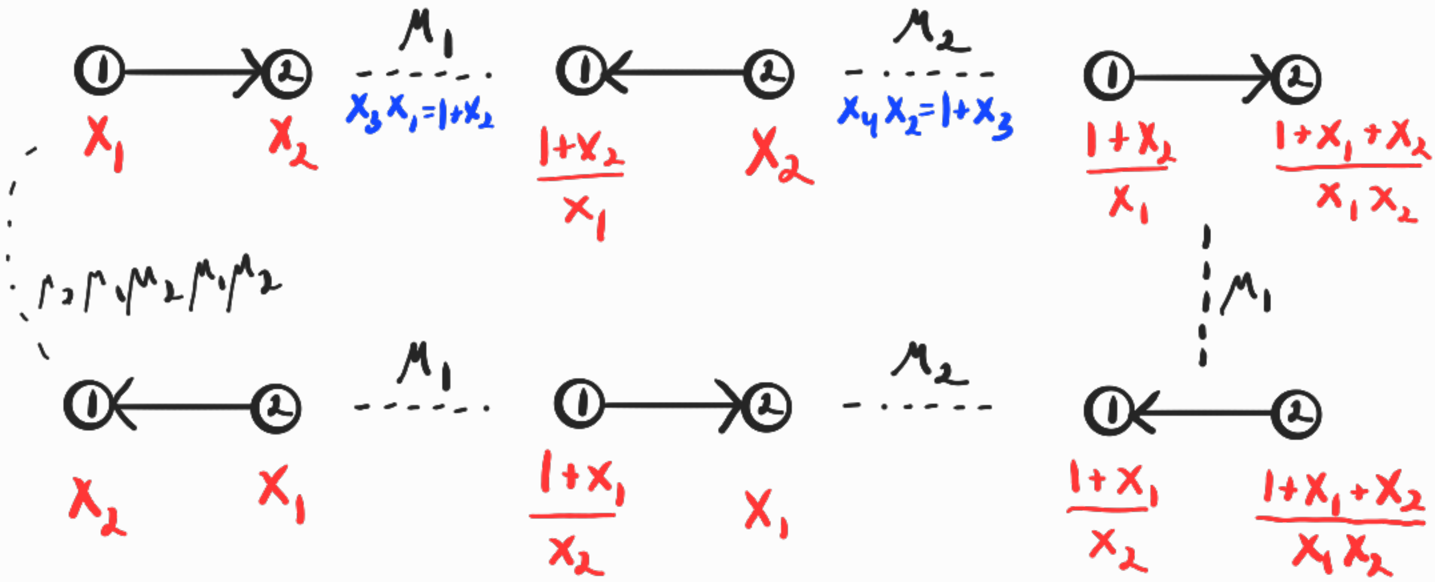
$\underline{x} = (x_1, x_2, x_3)$ $\qquad\qquad$ $\underline{x}' = \left(x_1, \frac{x_1 + x_3^2}{x_2}, x_3\right)$

Let $\mathcal{X}$ be the set of all cluster variables obtained by mutating the seed $(\underline{x}, Q)$ (over all possible mutation sequences)

Defn: Let the ground ring be $R := \mathbb{C}[x_{m+1}, \dots, x_n]$. The <u>cluster algebra</u> $\mathcal{A}(\underline{x}, Q)$ is $R[\mathcal{X}]$.

The rank is the rank of Q

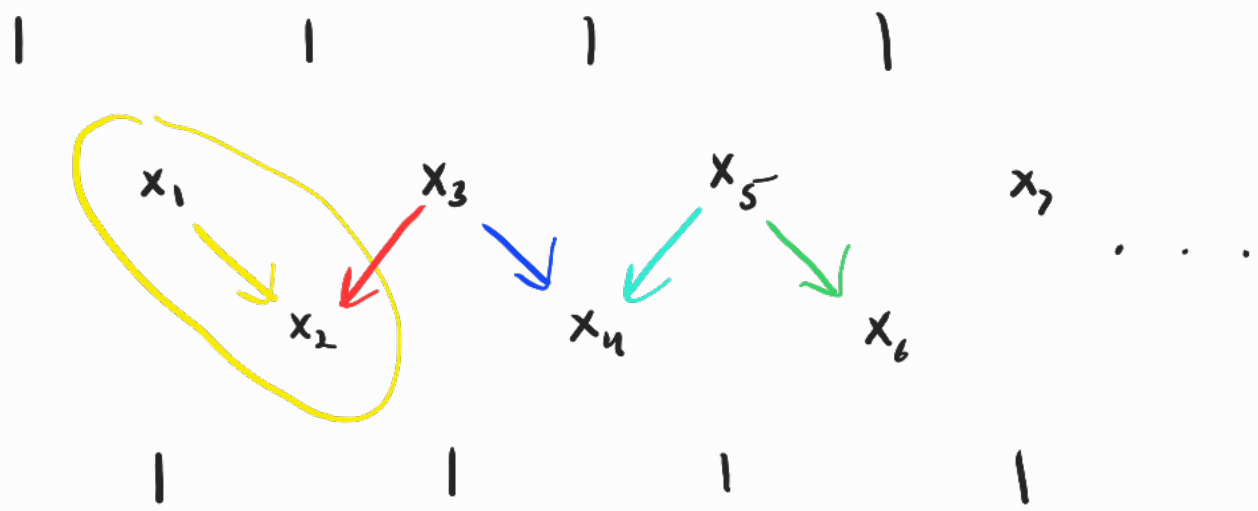
Example 5: Cluster algebra of type A_2



So $\mathcal{A}((x_1,x_2), ① \to ②) = \mathbb{Z}\left[x_1, x_2, \frac{1+x_2}{x_1}, \frac{1+x_1+x_2}{x_1x_2}, \frac{1+x_1}{x_2}\right]$

Rem: Usually # quivers and # cluster variables obtained by mutation is infinite.

Example Frieze $\mathbb{F}$



$$x_1x_3 = 1+x_2 \Rightarrow x_3 = \frac{1+x_2}{x_1}$$

$$x_4x_2 = 1+x_3 \Rightarrow x_4 = \frac{1+x_1+x_2}{x_1x_2}$$

$$x_5 x_3 = 1 + x_4 \Rightarrow x_5 = \frac{1+x_1}{x_2}$$

$$x_6 x_4 = 1 + x_5 \Rightarrow x_6 = x_2$$

$$x_7 x_5 = 1 + x_6 \Rightarrow x_7 = x_1$$

$$\vdots$$

So all elements of $\mathbb{F}$ are in the cluster algebra

$$\mathcal{A}((x_1, x_2), 1 \longrightarrow 2)$$